Depth-based methods for Functional Data Analysis

PhD in Business and Quantitative Methods

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Advisor: Raúl Jiménez

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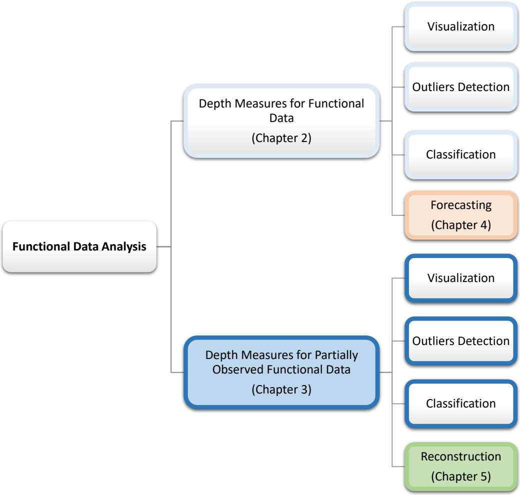
uc3m | Universidad Carlos III de Madrid

Thesis structure and contributions



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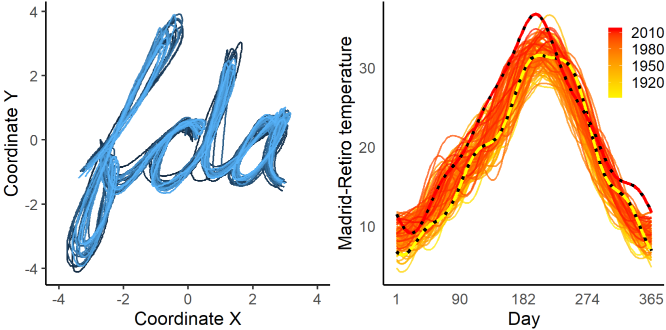
- **Depth-based methods** has depth measures as backbones.

Depth-based methods for **Functional Data Analysis**

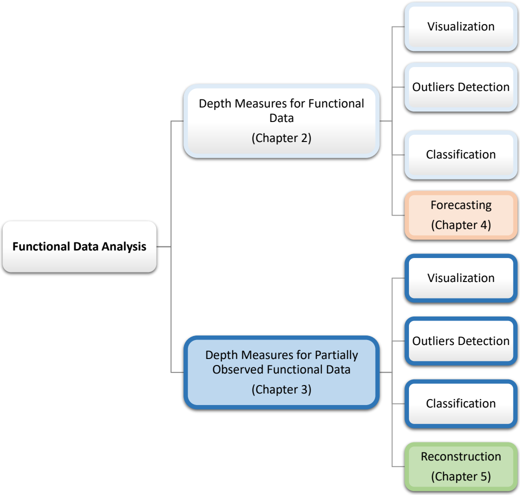


$$\Omega = \mathcal{C}([a, b])$$

$X_1, \ldots, X_n$ are n independent realizations of $X \sim P$

$$[a, b] \to \mathbb{R}$$

Chapter 2


Functional Data Analysis
Depth Measures for Functional Data
(Chapter 2)
Visualization
Outliers Detection
Classification
Forecasting
(Chapter 4)
Depth Measures for Partially Observed Functional Data
(Chapter 3)
Visualization
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Reconstruction
(Chapter 5)

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  - Properties: Nieto-Reyes and Battey (2016) and Nagy et al. (2017).
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In general, two families of functional depths (Nagy et al., 2016)

1. **Integrated depth measures**
2. Non integrated depth measures

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$$\mathrm{IFD}_w(x,P) = \int_0^1 \mathrm{D}(x(t),P_t)w(t)dt.$$

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Proposition FM depth and MBD are particular cases of the IFDϕ.

Being $\phi : [0,1] \to [0,1]$, IFDs of the form

$$\mathrm{IFD}_\phi(x, P) = \int_0^1 \phi(P_t(x(t))dt,$$

with $\phi_{FM}(y) = 1 - \left|\frac{1}{2} - y\right|$ and $\phi^j_{MBD}(y) = 1 - y^j - (1-y)^j$.

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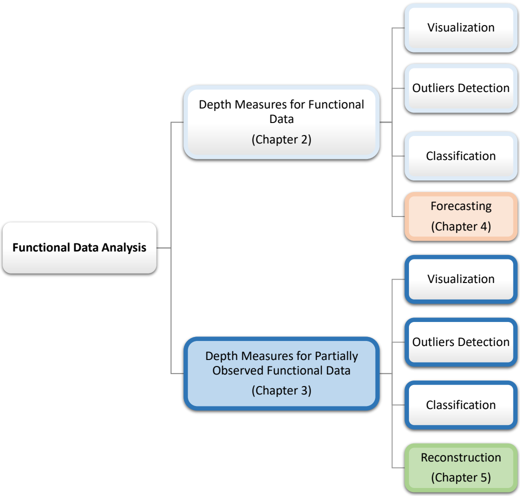
Chapter 3



Elías, A., Jiménez, R., Paganoni, A., Sangalli, L. (2019). A Depth for Censured Functional Data. Universidad Carlos III de Madrid, Departamento de Estadística.

Elías, A., Jiménez, R., Paganoni, A. and Sangalli, L. (2020). Integrated Depths for Partially Observed Functional Data, *(Submitted)*.

Depth Measures for Partially Observed Functional Data

1 Partially Observed Functional Data (POFD)

2 Integrated depth measures for POFD

3 Simulation results

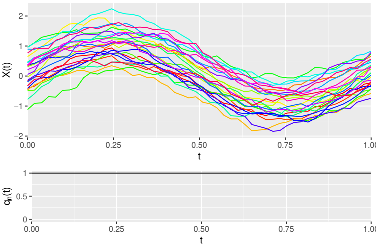
4 Case studies

4.1 Outlier detection for POFD

4.2 Depth-to-Depth classifiers for POFD

Partially Observed Functional Data (POFD)

Functional Data Analysis



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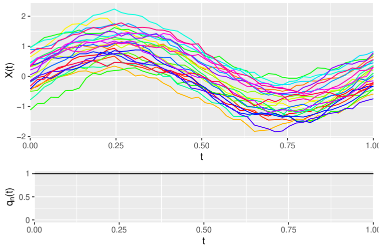

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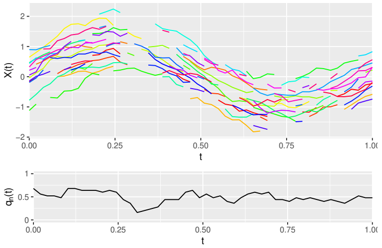

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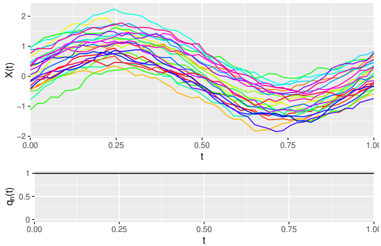

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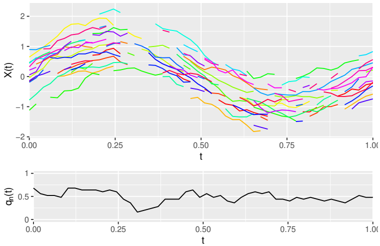

$$\text{Observability} : \mathcal{O}_1, \ldots, \mathcal{O}_n \sim Q$$

$$(X_1, \mathcal{O}_1), \ldots, (X_n, \mathcal{O}_n) \sim P \times Q$$

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Three dimensional arrays reconstruction
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... hampering the application of many FDA techniques.

Integrated Depth for Partially Observed Functional Data

Definition Partially Observed Integrated Functional Depth (POIFD).

The POIFD of $(x, \mathcal{O})$ with respect to $P \times Q$ is

$$\text{POIFD}((x, \mathcal{O}), P \times Q) = \int_{\mathcal{O}} \text{D}(x(t), P_t) w_{\phi}(t|\mathcal{O}) dt.$$

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In particular, we consider the following weighting function restricted to $\mathcal{O}$:

$$w_\phi(t|\mathcal{O}) = \frac{\phi(Q(t))}{\int_{\mathcal{O}} \phi(Q(t)) dt}$$

being $Q(t) = \mathbb{P}(\mathcal{O} \ni t)$, and ϕ a non-decreasing continuous function.

Integrated Depth for Partially Observed Functional Data

Theorem (POIFD is a IFD, in expectation)

Under the Missing-Completely-At-Random assumption,

$$\mathbb{E}[\mathrm{POIFD}((x,\mathcal{O}), P \times Q)] = \mathrm{IFD}_w(x, P),$$

with $w(t) = \mathbb{E}\left[1_{\mathcal{O}}(t) w_\phi(t|\mathcal{O})\right]$.

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Theorem (Consistency)

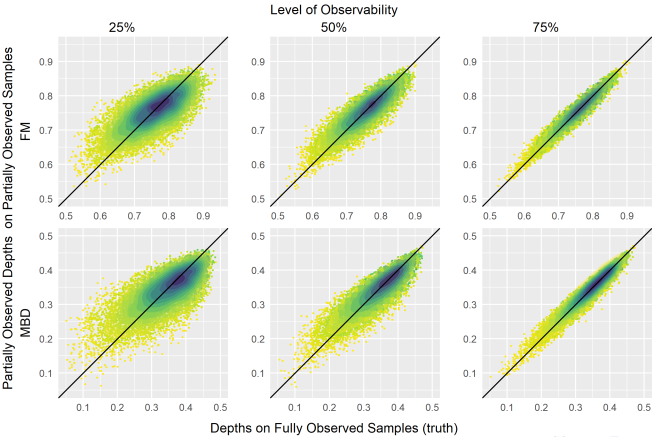
If the univariate depth D *satisfies the properties, then*

$$\lim_{n\to\infty} \lim_{T\to\infty} \mathrm{POIFD}_T((x, \mathcal{O}), P_n \times Q_n) = \mathrm{POIFD}((x, \mathcal{O}), P \times Q) \quad \textit{almost surely.}$$

Simulation Results

Highlights

1. The empirical POIFD agrees with the empirical IFD (unreachable in practise).



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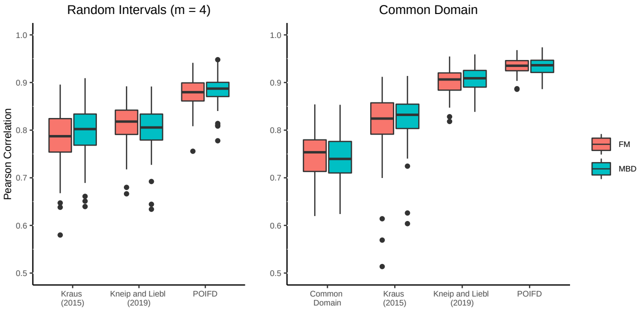

Simulation Results

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1. The empirical POIFD agrees with the empirical IFD (unreachable in practise).
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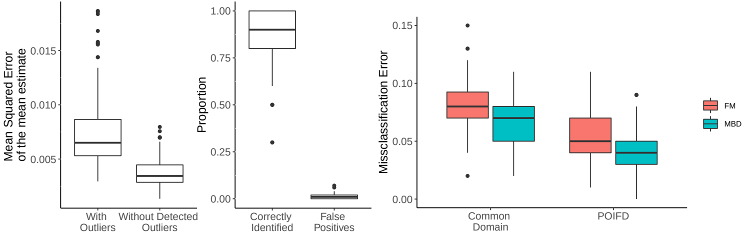
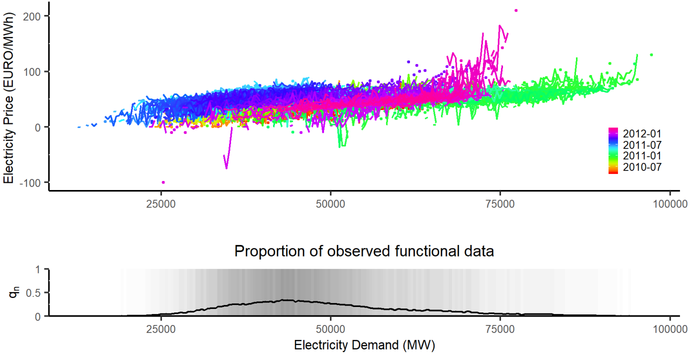
3. POIFD provides Functional Boxplot, Outliergram and DDplot for POFD, with outlier unmasking and classification capabilities.



German Electricity Supply Functions



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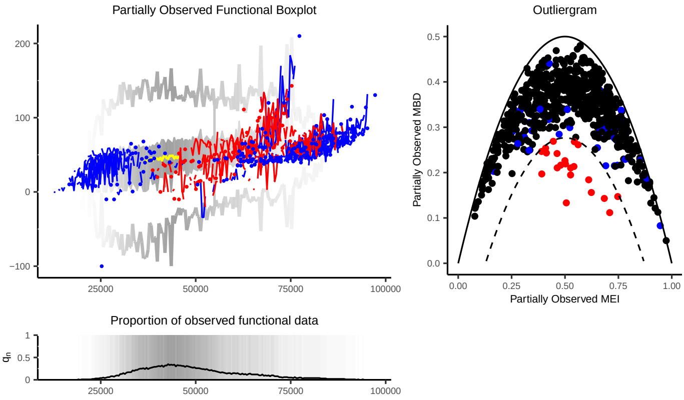

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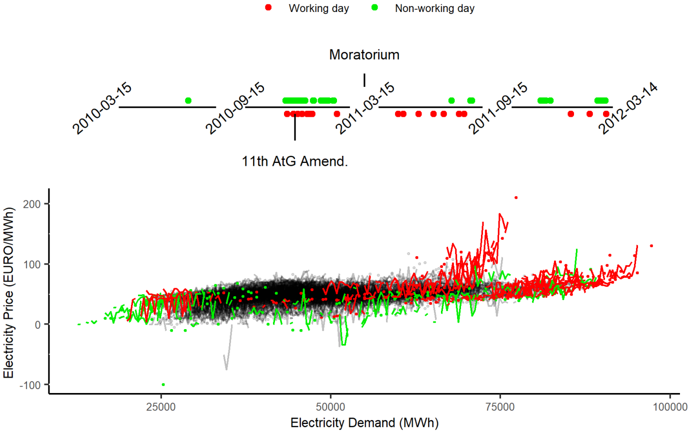
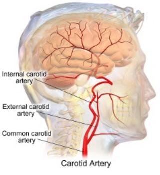
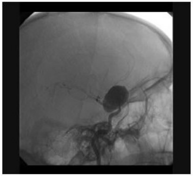
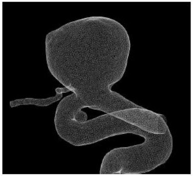


Aneurisk65 Project







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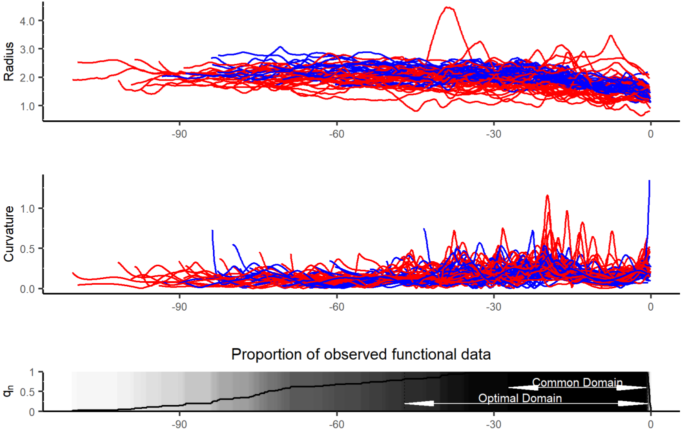
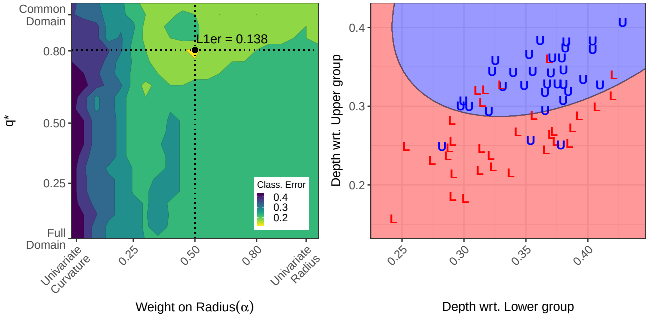


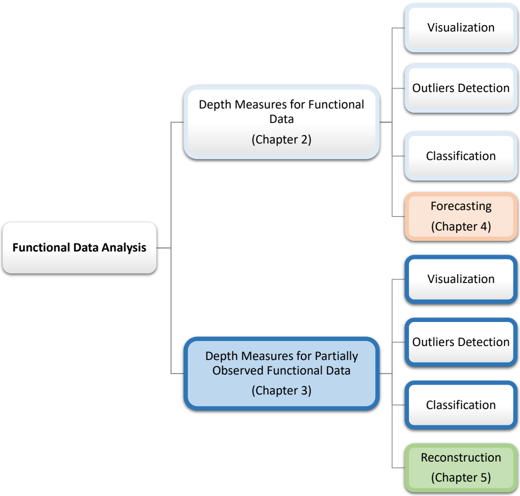
Aneurisk65 Project

Classification of Upper and Lower patients

- Multivariate functional POFD (MBD) measure (Ieva and Paganoni, 2013).
- Weighting function ϕ is a continuous approximation to $qH(q - q^*)$, being H the Heaviside step function and q^* a small nonnegative threshold.



Chapter 4



Elías, A. and Jiménez, R. (2017). Prediction Bands for Functional Data Based on Depth Measures. Universidad Carlos III de Madrid, Departamento de Estadística.

Elías, A. and Jiménez, R. (2018). A Depth-based Method for Functional Time Series Forecasting. *arXiv:1806.11032*.

Depth-based method for Functional Time Series Forecasting

1 Functional Time Series Forecasting

2 Depth-based forecasting method

3 Simulation results

4 Case studies

   4.1 Spanish Electricity Demand

   4.2 NOx Emission in Plaza España

About functional time series forecasting

Time series where $\{y_k, k \in \mathbb{N}\}$ where each y_k is a random function $t \to y_k(t)$, $t \in [a, b]$ (Hörmann and Kokoszka, 2012).

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- Data-driven and interpretable.

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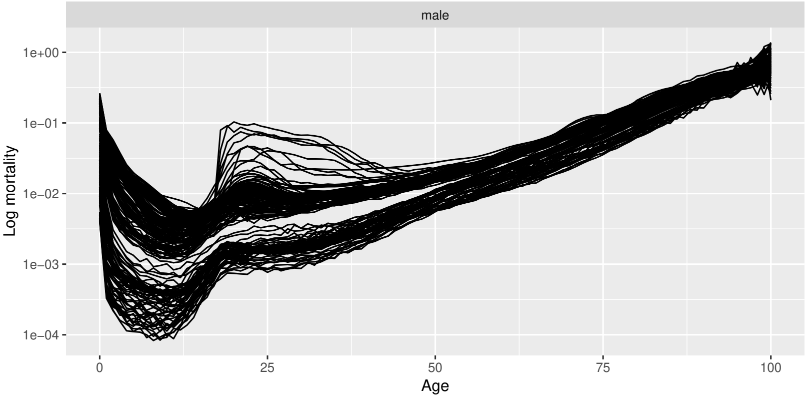

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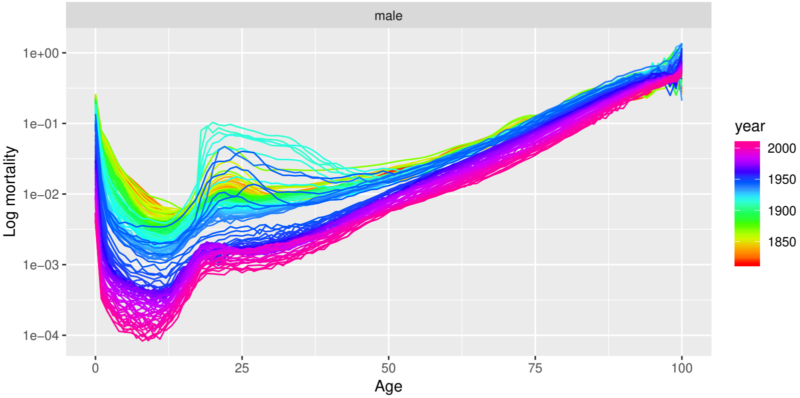

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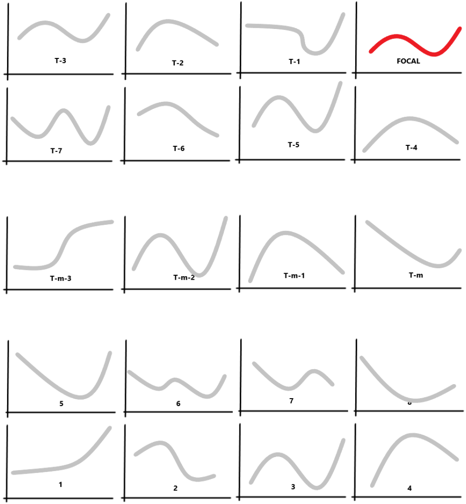
Focal central region (envelope)



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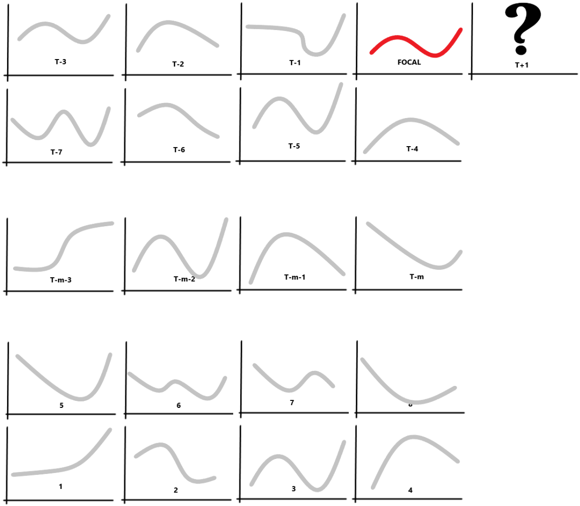

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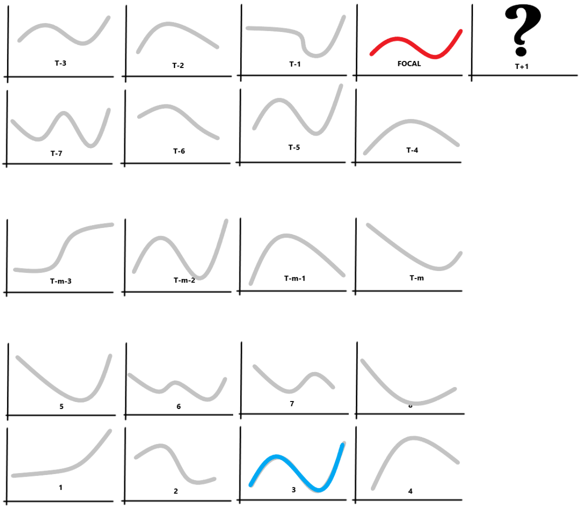

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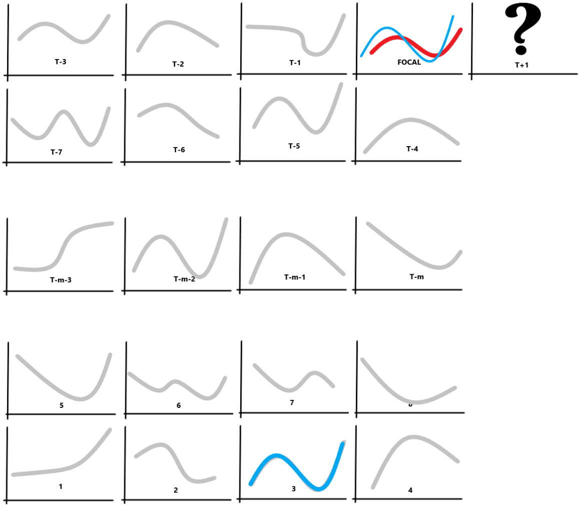

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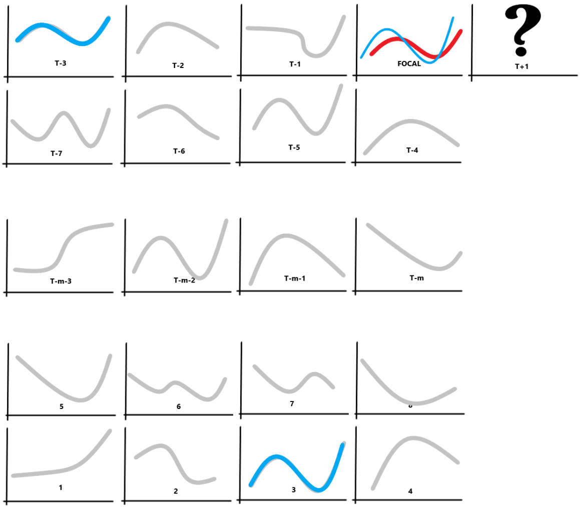

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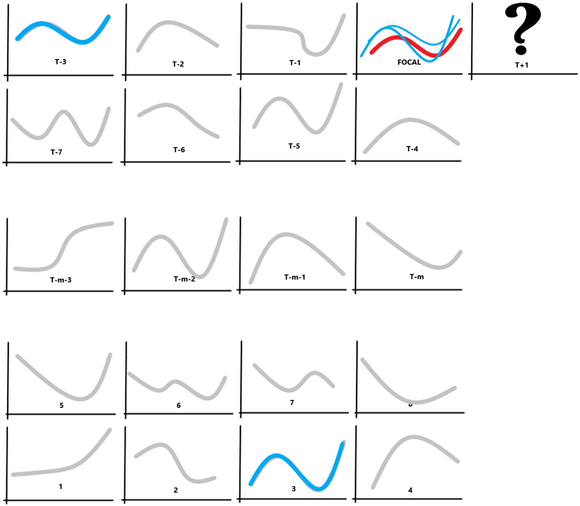

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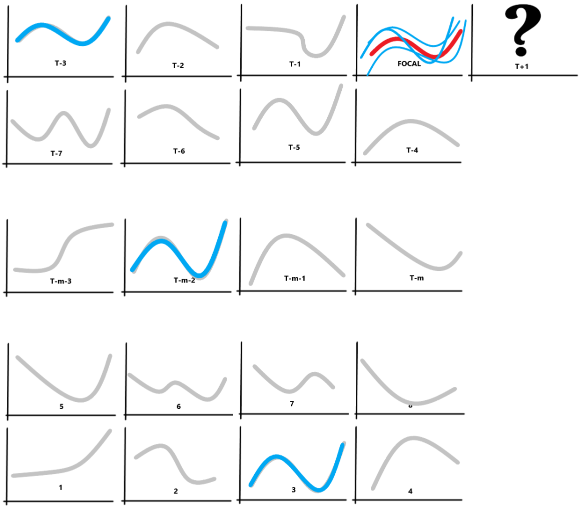

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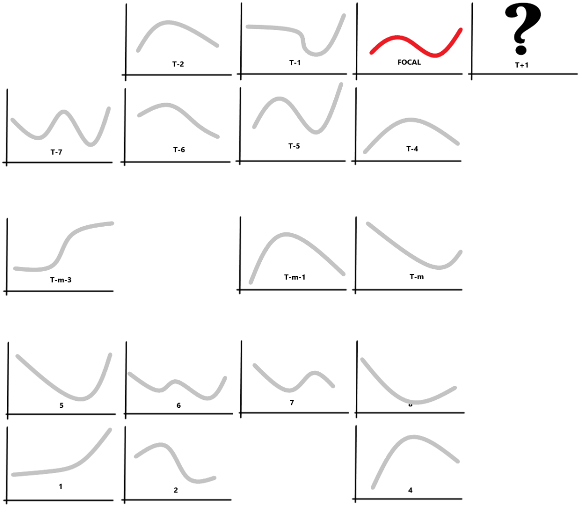

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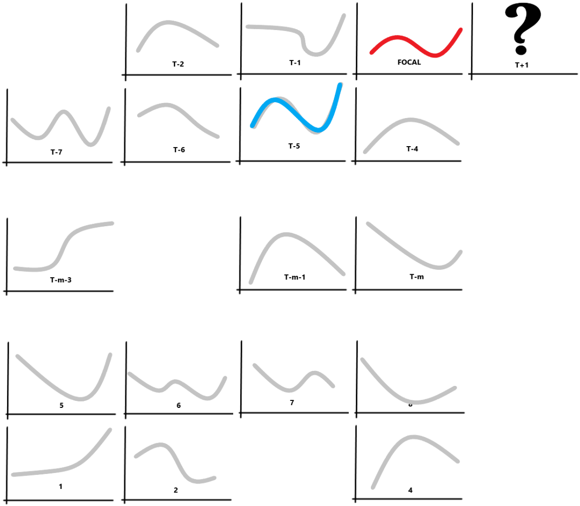

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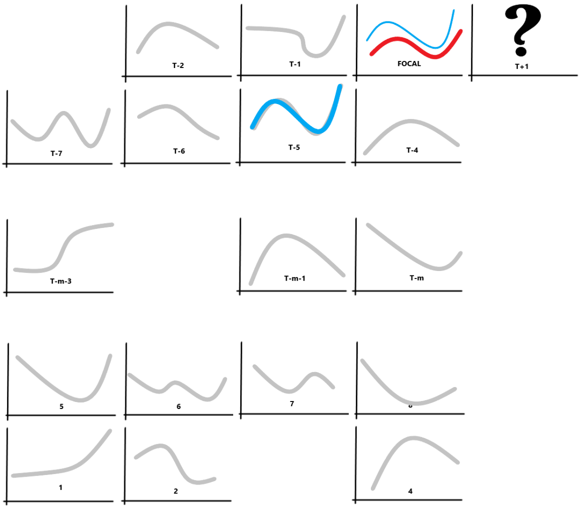

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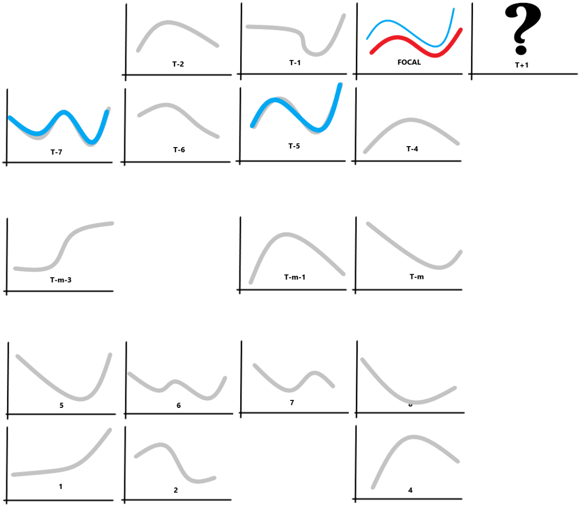

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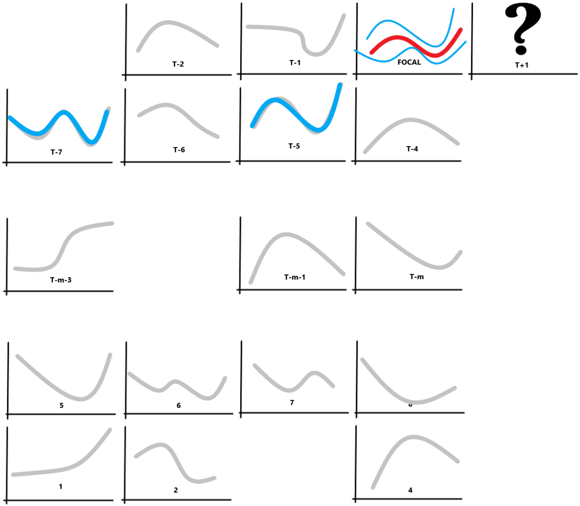

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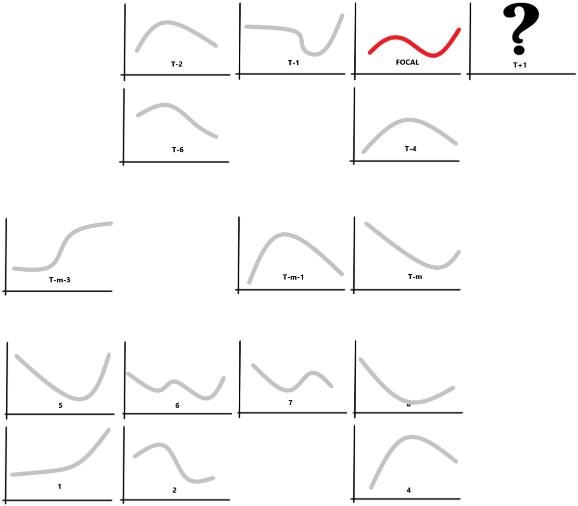

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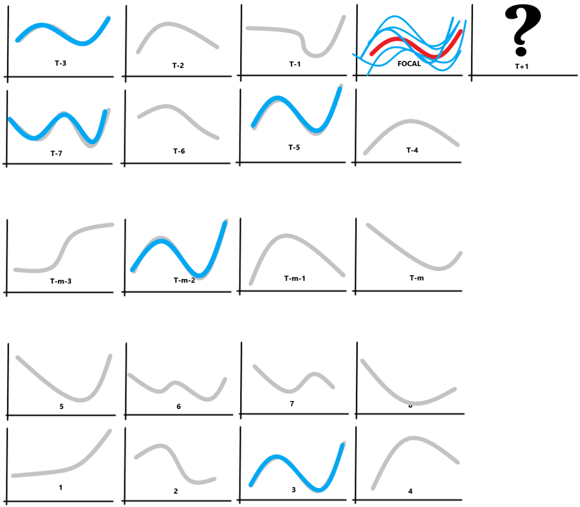

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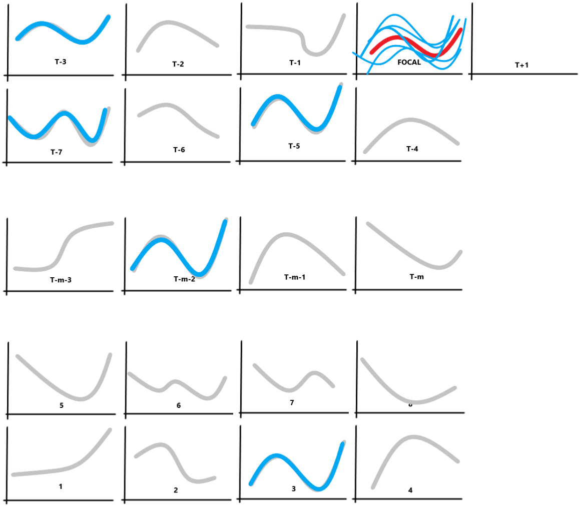

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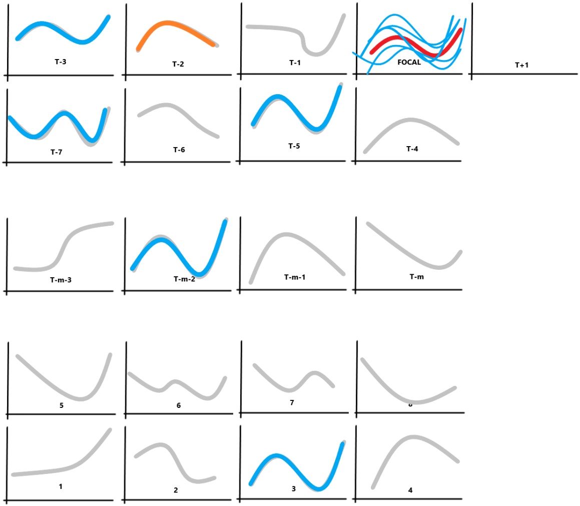

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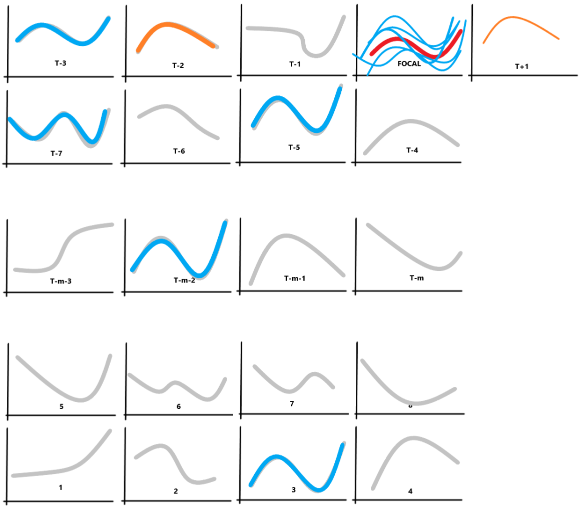

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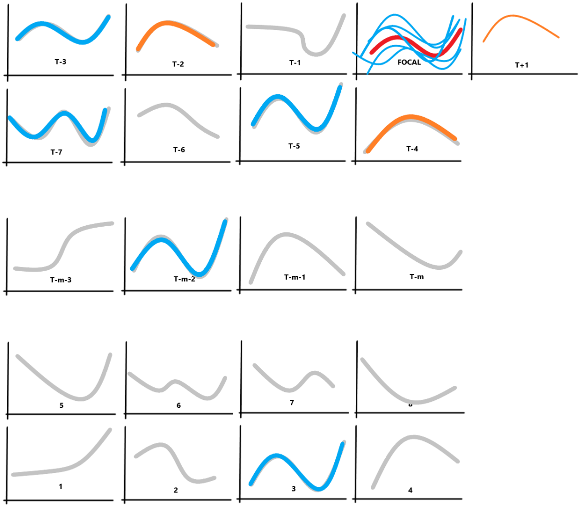

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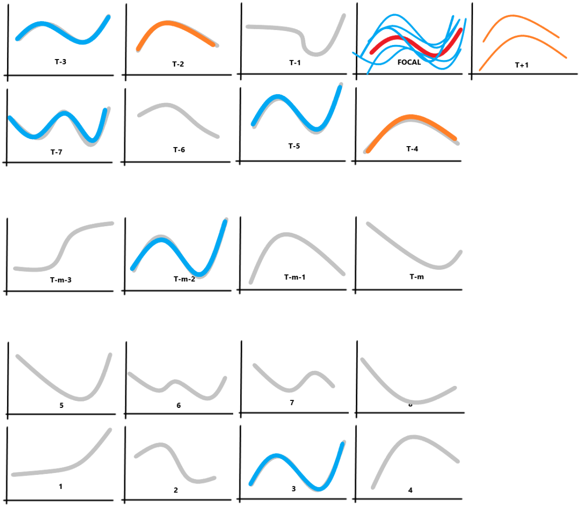

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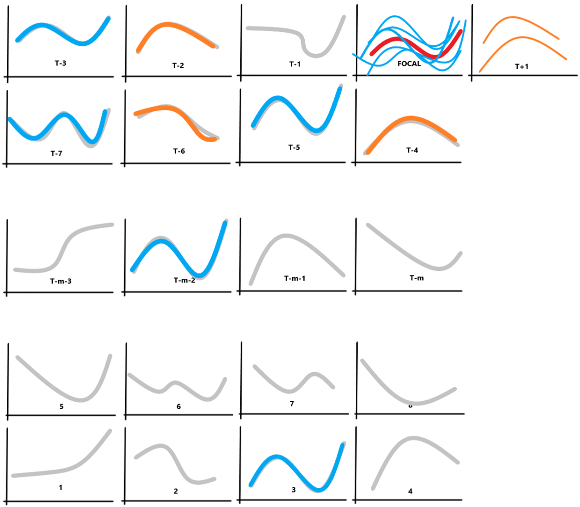

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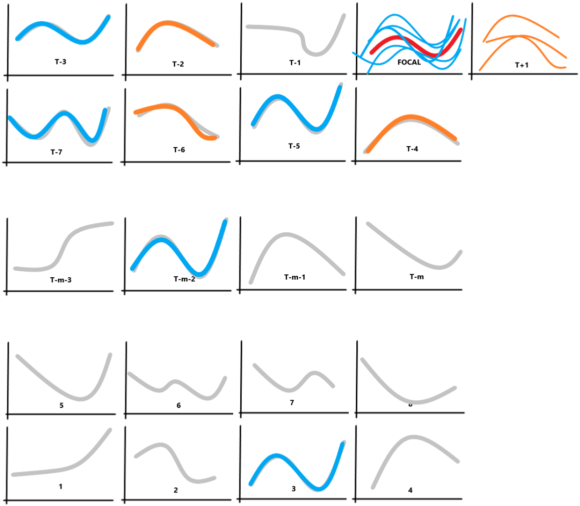

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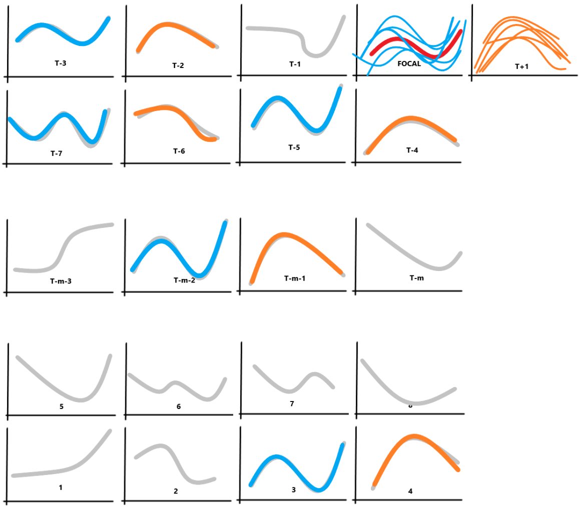

Focal central region (envelope)

The *Focal Central Region* (FCR) delimited by $\mathcal{J}_k$ is

$$R(\mathcal{J}_k) = \left\{ (t, y(t)) : t \in D_f, \min_{x \in \mathcal{J}_k} x(t) \leq y(t) \leq \max_{x \in \mathcal{J}_k} x(t) \right\}.$$

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We look for a set $\mathcal{J}$ of past focal curves such that:

1. f is deep in $\mathcal{J} \cup \{f\}$, the deepest if possible.
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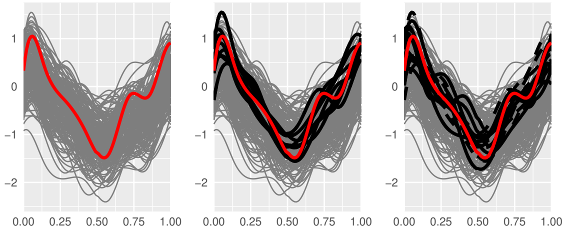

The prediction method

Point forecast

Being $\mathcal{E}y$ the *extension of* y.

$$\hat{p}_\theta = \frac{\sum_{y \in \mathcal{J}} w_y \mathcal{E}y}{\sum_{y \in \mathcal{J}} w_y}, \quad \text{with} \quad w_y = \exp(-\theta \|y - f\|^2/\delta),$$

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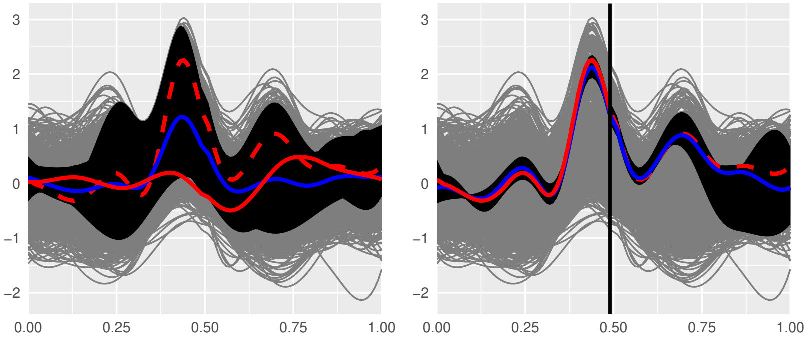
Parameter selection

- θ is selected by minimizing the MSE.
- Given a mean coverage, k is selected by minimizing the Winkler Score.

The prediction method

Figure shows an example of point and band prediction based on 1000 curves from a simulated FTS. Both one-step ahead forecasting and dynamic updating are illustrated.



Simulation Results

Highlights

* Aue, A., Norinho, D., and Hörmann, S. (2015). On the prediction of stationary functional time series. *Journal of the American Statistical Association*.

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- Best performance in **short-term** predictions.
- Conclusions remain for small (200) and large sample sizes (1000).

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Spanish Electricity Demand

- Demand in megawatts (MW) from January first 2014 to December 31st 2018 each 10 minutes.
- Exercise:
  - Forecasting the whole year 2018, day by day and half days.
  - Without any preprocessing.

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			D-BF			$D\text{-}BU_{0.5}$		
	Average	FPCF	30	90	365	30	90	365
MAPE	8.86	6.58	2.66	2.60	2.54	1.54	1.54	1.54

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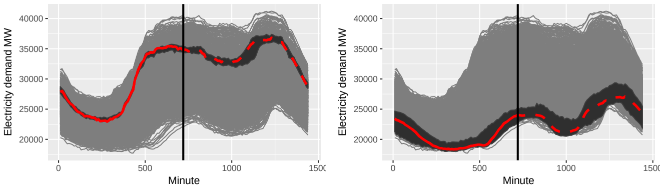
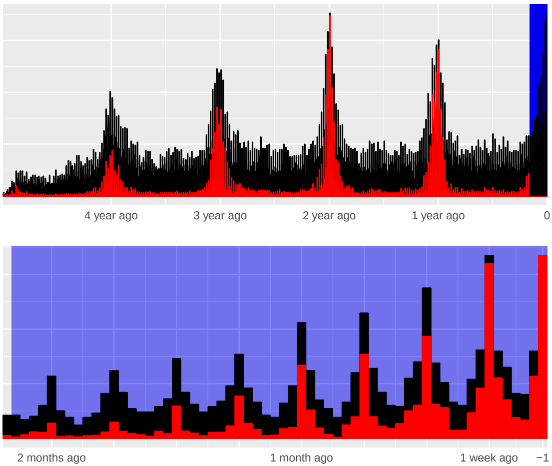
10-**deepest functions for 29th November and 25th December.**



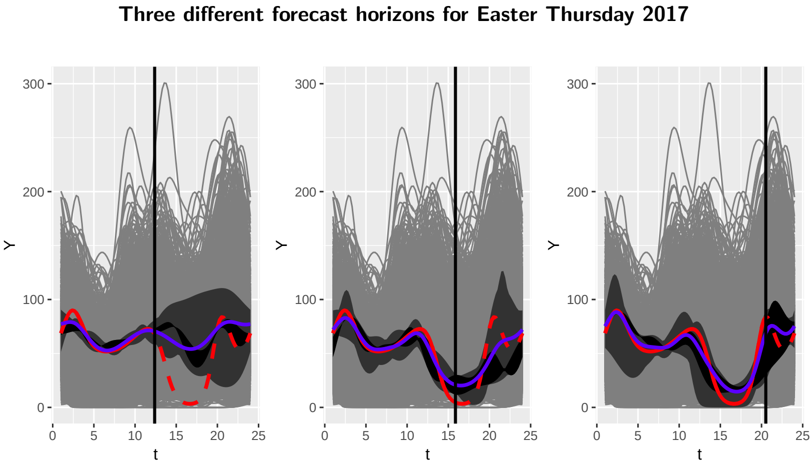
$\mathcal{J}(f_{tuesday}/29/11/2016) = \{$tuesday/26/01/2016, friday/25/11/2016, wednesday/25/11/2015, wednesday/27/01/2016, wednesday/02/12/2015, tuesday/25/02/2014, wednesday/17/12/2014, wednesday/15/01/2014, friday/29/01/2016, wednesday/10/02/2016 $\}$.
$\mathcal{J}(f_{sunday}/25/12/2016) = \{$thursday/25/12/2014, sunday/02/11/2014, friday/25/12/2015, sunday/01/11/2015, wednesday/01/01/2014$\}$.

Spanish Electricity Demand

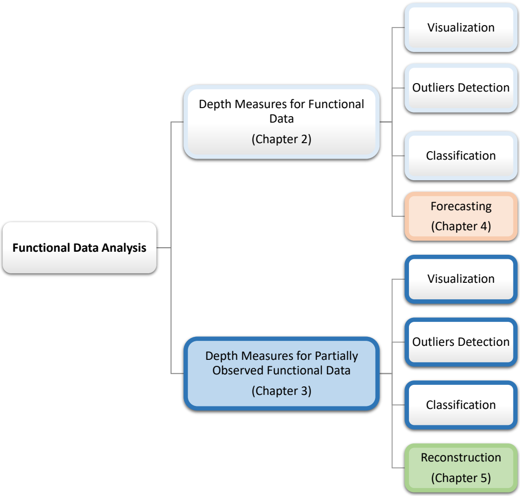
Daily temporal dependency visualization



NOx Emissions



Chapter 5



Elías, A., Jiménez, R. and Shang, H. (2020). Depth-based reconstruction method for partially observed functional data, (*working process*).

Depth-based Method for partially observed functional data reconstruction

1. Partially observed functional data reconstruction
2. Depth-based reconstruction method
3. Simulation results
4. Case studies
    4.1. Spanish daily maximum temperatures by weather station
    4.2. Japanese age-specific mortality by prefecture

Partially observed functional data reconstruction

The problem of reconstruction

- The observed and missing parts of x_i are

$$\begin{cases} (x, \mathcal{O}), \\ (x, m) \text{ with } M = [a, b] \setminus \mathcal{O}. \end{cases}$$

- The goal is to estimate each (x, m).

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The literature about reconstruction

- Functional linear ridge regression (Kraus, 2015).
- Local Linear Kernel (Kneip and Liebl, 2019).

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The problem of reconstruction

- The observed and missing parts of x_i are

$$\begin{cases} (x, \mathcal{O}), \\ (x, m) \text{ with } M = [a, b] \setminus \mathcal{O}. \end{cases}$$

- The goal is to estimate each (x, m).

The literature about reconstruction

- Functional linear ridge regression (Kraus, 2015).
- Local Linear Kernel (Kneip and Liebl, 2019).

Why another one?

- Covariance estimation issues.
  - **Estimation is not possible** if there are not complete functions.
  - **Hard to estimate** if the covariance structure is complex or samples are poorly observed.
- Fidelity to raw data, no preprocessing or postprocessing.

Kraus, D. (2015). Components and completion of partially observed functional data. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*.

Kneip, A. and Liebl, D. (2019). On the optimal reconstruction of partially observed functional data. *The Annals of Statistics*.

Depth-based reconstruction method

Envelope. Chapter 3 + Chapter 4.

Given a curve partially observed curve $(x_f, \mathcal{O}_f)$, we look for a set $\mathcal{J}$ such that:

1. $(x_f, \mathcal{O}_f)$ is deep in $\mathcal{J} \cup \{f\}$, the deepest if possible.
2. f and the domain $[a, b]$ is covered by $\mathcal{J}$ as much as possible.
3. f is surrounded by near curves of $\mathcal{J}$, as many as possible.

$+$

Point estimator

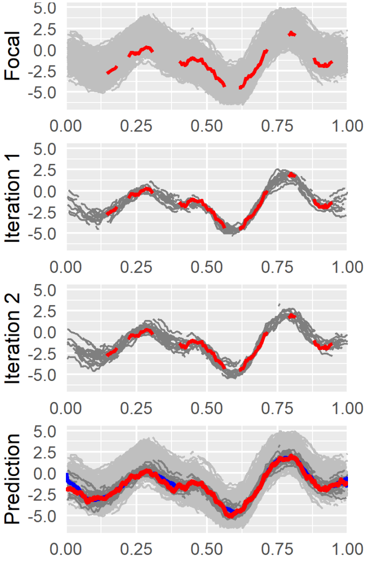
$$\hat{X}_i(t) = \frac{\sum_{j \in \mathcal{J}_i(t)} w_j X_j(t)}{\sum_{j \in \mathcal{J}_i(t)} w_j}, \quad \text{with} \quad w_j = \exp(-\theta \| (X_i, O_i) - (X_j, O_j) \| / \delta),$$

δ being the distance scale defined by $\delta = \min_{j \in \mathcal{J}_i} \| (X_i, O_i) - (X_j, O_j) \|$.

$$\theta^* = \operatorname{argmin} \mathbb{E} \| (X_i, O_i) - (\hat{X}_i, \hat{O}_i) \|^2,$$

with $\hat{O}_i = \cup_{j \in \mathcal{J}_i} (M_i \cap O_j)$.

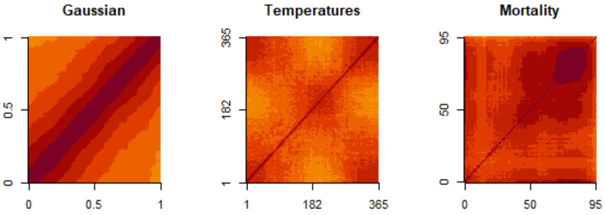
The method


Focal
Iteration 1
Iteration 2
Prediction
5.0
2.5
0.0
-2.5
-5.0
0.00
0.25
0.50
0.75
1.00

Simulation Results

Highlights

1. The depth-based method seems to be superior with complex covariance structures.



Simulation Results

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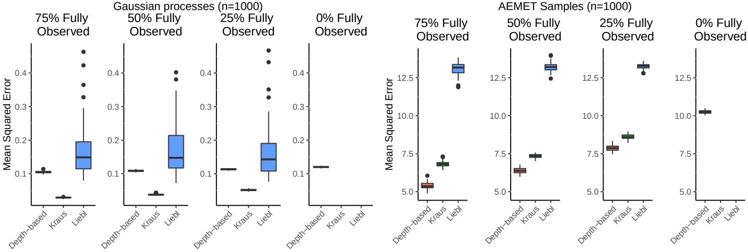
2. The depth-based method gets estimations even without complete functions.

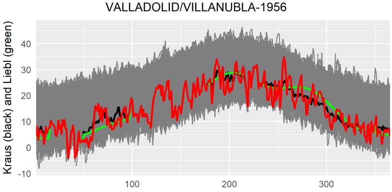


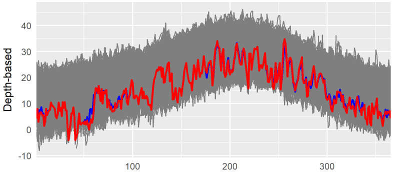
Simulation Results

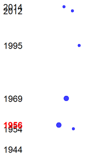
Highlights

1. The depth-based method seems to be superior with complex covariance structures.
2. The depth-based method gets estimations even without complete functions.
3. The competitors are preferable with smooth functions.

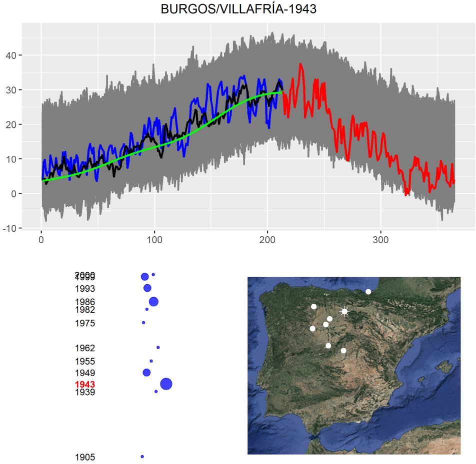
Simulation Results



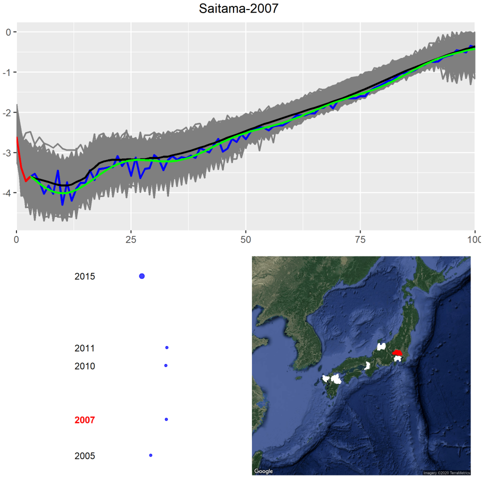




Spanish daily maximum temperatures by weather station



Japanese age-specific mortality by prefecture



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CODE `https://github.com/aefdz`.

Depth-based methods for Functional Data Analysis

PhD in Business and Quantitative Methods

Antonio Elías Fernández

Advisor: Raúl Jiménez

3th, April 2020

uc3m | Universidad Carlos III de Madrid

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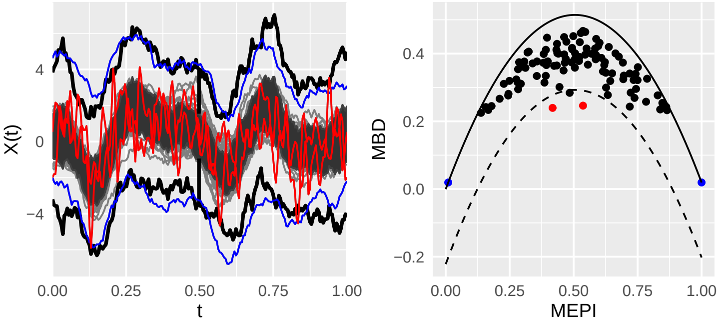

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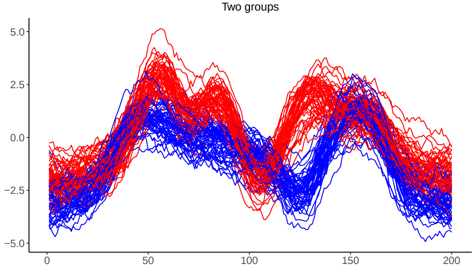
Populations Comparison and Classification

Depth-to-Depth Plot

Being $P_{G1}, P_{G2} \in \mathcal{P}_{\mathcal{C}([a,b])}$ and FD any functional depth,

$$\mathrm{DD}_X(P_{G1}, P_{G2}) = \{(\mathrm{FD}(X, P_{G1}), \mathrm{FD}(X, P_{G2})) \qquad \forall X \in \mathcal{P}_{\mathcal{C}([0,1])}\}.$$



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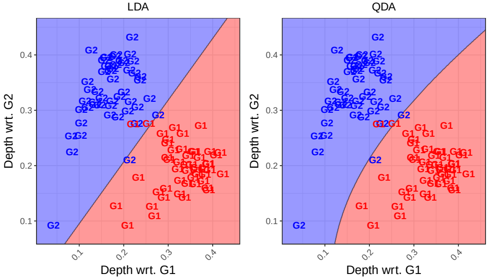

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